Malaysian Journal of Computer Science, Vol. 9 No. 1, June 1996, pp. 42-48

CLASSES OF INTERVAL-VALUED FUZZY RELATIONAL INFERENCE TEMPLATES

Yew Kok Meng
Faculty of Computer Science & Information Technology
University of Malaya
50603 Kuala Lumpur
Malaysia
email: yew@fsktm.um.my.edu

L. J Kohout
Department of Computer Science
Florida State University
Florida, USA
email: kohout@xi.cs.fsu.edu

ABSTRACT

In this paper, we present the various classes of inference templates which can be instantiated for making inferences in a fuzzy relational inference scheme. These templates are interval-valued and they capture the upper and lower bounds of fuzzy inference using point data. We present five classes of interval-valued inference templates which can be instantiated with bounded fuzzy connectives.

***Keywords*: *Knowledge-based systems, Interval-valued Inference, Relational Products*.**

1.0 INTRODUCTION

Inference techniques had traditionally been based on modus ponens, modus tollens, syllogism and contraposition. When fuzzy if-then rules were introduced, modus ponens were generalized [1, 2, 3, 4] and were correspondingly called generalized modus ponens. Zadeh suggested the compositional rule of inference for this type of fuzzy conditional inference [1]. Other authors [5, 6, 7, 8] have suggested different methods and also investigated generalized modus tollens, syllogism and contraposition.

In this paper, we look at inference based on subset containment, supported by the power set theory and mathematical relations. This method was first proposed in 1977 by Bandler and Kohout [9, 10, 11, 12]. They introduced the relational products for the composition of relations and provided both the harsh and the mean set-theoretic definitions of the products based on the power set theory [12]. Later, in 1993 DeBaets and Kerre [13] introduced modifications to these relational products to incorporate the non-emptiness conditions and provided two alternative set-theoretic definitions to the relational products. These are based on the theory of t-norms and t-conorms. We shall refer to the original products as the BK- products and the later modifications as the B- and K-products.

One crucial factor in using the relational products for inferences is the choice of the fuzzy connectives. These connectives play a critical role in the field of fuzzy systems [14]. T-norms and t-conorms had been used for the AND and OR connective in the B- and K- logic-based relational product definitions as opposed to the traditional MAX and MIN counterpart in the BK- relational products. However, there are an infinite number of t-norms and t-conorms that could be used and it is difficult to know which is the more suitable. To avoid this situation, interval-valued approximate inference templates were abstracted from these relational products definitions.

The type of inference studied here belongs to a category called interval-valued inference as opposed to point-based inference and is based on the meta theory of the *checklist paradigm* [11, 15, 16]. Using point data, the inference structures instantiated from the templates capture the upper and the lower bounds of the fuzzy inference within which the computation must lie. We introduce in this paper five different classes of interval-valued inference templates corresponding to the five different measures of the fuzzy implications [11, 15, 16]. Each class provide inference templates for the upper and the lower bounds of fuzzy inference.

2.0 RELATIONAL PRODUCTS

Given the sets A, B and C and the relations $R \in \Re(A \times B)$ and $S \in \Re(B \times C)$, we can compose new relations $T \in \Re(A \times C)$ in four different ways. These different methods of composition are called circle product, sub-triangle product, super-triangle product and the square product [11, 12]. They each have different meanings and are defined as follows:

Circle product

$$(R \circ S)_{ik} = \max_j \left(\min(R_{ij}, S_{jk})\right)$$

Sub-Triangle Product

$$(R < S)_{ik} = \min_j \left(R_{ij} \rightarrow S_{jk}\right)$$

Super-Triangle Product

$$(R < S)_{ik} = \min_{j} (R_{ij} \leftarrow S_{jk})$$

Square Product

$$(R \quad S)_{ik} = \min_{j} (R_{ij} \leftrightarrow S_{jk})$$

where $a \in A$, $b \in B$ and $c \in C$, the connective $\rightarrow$ is the implication operator and $a \leftrightarrow b$ is the equivalence operator. R_{ij} is the element in the i^{th} and the j^{th} column of the relation R in the matrix notation [17]. The above relational products are harsh definitions. The equivalent mean definitions [11] are as follows:

$$(R < S)_{ik} = \frac{1}{n}\sum_{j=1}^{n}(R_{ij} \rightarrow S_{jk})$$

$$(R > S)_{ik} = \frac{1}{n}\sum_{j=1}^{n}(R_{ij} \leftarrow S_{jk})$$

$$(R \quad S)_{ik} = \frac{1}{n}\sum_{j=1}^{n}(R_{ij} \leftrightarrow S_{jk})$$

Other modifications of the mean relational products above can be found in [18].

3.0 ALTERNATIVE RELATIONAL PRODUCTS DEFINITIONS

Recently, DeBaets and Kerre [13] noticed that the definitions of the above BK- relational products (sub-triangle, super-triangle and the square products) did not include the non-emptiness conditions and showed that surprising results may be inferred using the harsh criterion. They revised the definitions of the relational products and suggested two alternative definitions based on the theory of t-norms [14, 19, 20, 21, 22]. Much of the work published on the properties of these alternative relational product definitions were based on the harsh criterion. However, this criterion is known to be inappropriate in situations where individual inferential 'evidences' were to be aggregated. Under such circumstances, the mean criterion would be more appropriate. In one medical knowledge based system *Clinaid* [23], the mean criterion was used for its inference structures. DeBaets and Kerre provided only the alternative (harsh) set-theoretic definitions for the above relational products as follows:

Definitions 1

1.1 The B- sub-triangle relational product:

$$(R \triangleleft^{*} S)_{ik} = \min(\inf_{j}(R_{ij} \rightarrow S_{jk}), \sup_{j} R_{ij}, \sup_{j} S_{jk})$$

which is equivalent to

$$\min((R <_{bk} S)_{ik}, \max_{j} R_{ij}, \max_{j} S_{jk})$$

for a finite fuzzy set using the harsh criterion where $<_{bk}$ is the original definition of Bandler and Kohout.

1.2 The B- super-triangle relational product:

$$(R \triangleright^{*} S)_{ik} = \min(\inf_{j}(R_{ij} \leftarrow S_{jk}), \sup_{j} R_{ij}, \sup_{j} S_{jk})$$

1.3 The B- square-triangle relational product :

$$(R \square^{*} S)_{ik} = \min(\inf_{j}(\xi(S_{jk}, R_{ij})), \sup_{j} R_{ij}, \sup_{j} S_{jk})$$

where $\xi(x,y) = \Im((x \rightarrow y), (x \leftarrow y))$ is the fuzzy equivalence operator.

Definitions 2

2.1 The K- sub-triangle relational product:

$$(R \triangleleft_{k} S)_{ik} = \min(\inf_{j}(R_{ij} \rightarrow S_{jk}), \sup_{j} \Im(R_{ij}, S_{jk}))$$

which is equivalent to

$$\min((R <_{bk} S)_{ik}, \max_{j} \Im(R_{ij}, S_{jk}))$$

for a finite fuzzy set using the harsh criterion where $\Im$ is the triangular norm.

2.2 The K- super-triangle relational product:

$$(R \triangleright_{k} S)_{ik} = \min(\inf_{j}(R_{ij} \leftarrow S_{jk}), \sup_{j} \Im(R_{ij}, S_{jk}))$$

2.3 The K- square-triangle relational product:

$$(R \boxdot S)_{ik} = \min(\inf_j (\xi (S_{jk}, R_{ij}), \sup_j \Im(R_{ij}, S_{jk})))$$

4.0 FUZZY RELATIONAL INFERENCE

Let us illustrate the meanings of the above four BK-relational products with a simplified document retrieval example. Suppose U be the set of users, T be the set of terms, D be the set of documents and the fuzzy relations R and S are represented as 2 slot open sentences as follows:

R : User ___ is interested in documents that deal with the term ___ to degree ___

S : The term ____ is dealt with in document ___ to degree____.

The relational products provide new relations from set User to the set Document and they can be used to infer the following:

$(R \text{ o } S)_{ik}$ gives the degree to which there is at least one term that the user U_i is interested in and that is dealt with in document D_k

$(R < S)_{ik}$ gives the degree to which the set of terms specified by user U_i are among the set of terms dealt with in document D_k

$(R > S)_{ik}$ gives the degree to which the set of terms specified by user U_i includes the set of terms dealt with in document D_k

$(R \quad S)_{ik}$ gives the degree to which the set of terms specified by user U_i are exactly equal to the set of terms dealt with in document D_k.

The human reasoning process relies on the aggregation of individual pieces of evidence to form a final decision. These individual pieces of evidence may be thought of as corresponding to the individual implications in the relational products. Bandler and Kohout [11] first proposed two methods of aggregating the implications: by the harsh and the arithmetic mean criterion. The former is not so useful where aggregation of evidence is crucial since the MIN connective does not compensate other evidences. The latter can be viewed as the logical connective half-way between conjunction and disjunction. Therefore, in most practical applications the mean criterion was used.

5.0 ABSTRACT INFERENCE TEMPLATES

Based on the above BK-, B- and K- sub, super triangle and square relational products definitions, the theory of t-norms and t-conorms and averaging operators [14, 24, 25, 26, 27, 28], we derive the following BK-, B- and K-abstract inference templates [29]. In the following abstractions or inference templates, the symbols $\circledvee$-$\circledwedge$ were used. The subscripts, if any, in these symbols refer to the position of the connectives in the syntactic form of the formula.

In this abstraction, we require the mean circle product and an abstract mean circle product template is defined as follows [29]:

$$(R \text{ o } S)_{ik} = \frac{1}{n} \sum_{j=1}^{n} (\circledwedge (R_{ij}, S_{jk}))$$

We provide below the BK-, B- and the K- list of abstract inference templates which must be instantiated before they can be used as inference structures in a fuzzy relational inference scheme. The method of instantiation of these inference templates are discussed in later sections.

The following are the *BK-abstract inference templates* corresponding to the sub, super and the square relational products.

SUB-BK template

$$(R <_{bk} S)_{ik} = \circledwedge_j (R_{ij} \rightarrow S_{jk})$$

SUPER-BK template

$$(R >_{bk} S)_{ik} = \circledwedge_j (R_{ij} \leftarrow S_{jk})$$

SQUARE-BK template

$$(R \square_{bk} S)_{ik} = \circledwedge_j (R_{ij} \leftrightarrow S_{jk})$$

The following are the *B-abstract inference templates* corresponding to the sub, super and the square relational products.

SUB-B template

$$(R \triangleleft S)_{ik} = \min(\circledwedge_{1\,j}(R_{ij} \rightarrow S_{jk}), \circledvee_{2\,j} R_{ij}, \circledvee_{3\,j} S_{jk})$$

SUPER-B template

$$(R \triangleright S)_{ik} = \min(\circledwedge_{1\,j}(R_{ij} \leftarrow S_{jk}), \circledvee_{2\,j} R_{ij}, \circledvee_{3\,j} S_{jk})$$

SQUARE-B template

$$(R \boxast S)_{ik} = \min(\underset{j}{\owedge_1}(R_{ij} \leftrightarrow S_{jk}),\ \underset{j}{\ovee_2} R_{ij},\ \underset{j}{\ovee_3} S_{jk})$$

The mean B (sub, super and square) inference templates were obtained by replacing $\owedge_1$ with $\frac{1}{n}\sum_{j=1}^{n}$

The following are the *K-abstract inference templates* corresponding to the sub, super and the square relational products.

SUB-K template

$$(R \triangleleft_k S)_{ik} = \underset{j}{\owedge_1}(\owedge_2(R_{ij} \rightarrow S_{jk}),\ \underset{j}{\ovee_3}(\owedge_4(R_{ij}, S_{jk})))$$

SUPER-K template

$$(R \triangleright_k S)_{ik} = \underset{j}{\owedge_1}(\owedge_2(R_{ij} \leftarrow S_{jk}),\ \underset{j}{\ovee_3}(\owedge_4(R_{ij}, S_{jk})))$$

SQUARE-K template

$$(R \boxdot_k S)_{ik} = \underset{j}{\owedge_1}(\owedge_2(R_{ij} \leftrightarrow S_{jk}),\ \underset{j}{\ovee_3}(\owedge_4(R_{ij}, S_{jk})))$$

The mean K inference templates were obtained by replacing $\owedge_2$ and $\ovee_3$ with $\frac{1}{n}\sum_{j=1}^{n}$.

All sixteen logical connectives have been shown to be bounded [15] and the above inference templates can be instantiated with these bounds., thus providing the inference structures. Denoting *ConTop* and *ConBot* as the upper and the lower bounds of the connective *Con*, the symbols $\ovee$-$\owedge$ in the above templates can be instantiated with the appropriate connectives in the set {AndTop, AndBot, Arithmetic mean, min, max, OrTop, OrBot} where AndBot(a,b) = max(0,a+b-1), AndTop(a,b) = min(a,b), OrBot(a,b) = min(1, a+b) and OrTop(a,b) = max(a,b).

6.0 CLASSES OF INTERVAL-VALUED INFERENCE TEMPLATES

In the above abstract templates, five different measures of the fuzzy implication $\rightarrow$ have been identified [11]. These measures were shown to be bounded and they lead to the five different classes of interval-valued inference templates corresponding to the sub-triangle relational products (SUB-BK, SUB-B, and the SUB-K inference templates). Ongoing research work is being done to identify the corresponding measures for the fuzzy connectives $\leftarrow$ and $\leftrightarrow$.

We provide below the lower and upper bounded inference templates of the five different classes of interval-valued inference templates for the BK-, B- and K- sub-triangle products with the harsh criterion and the mean criterion.

SUB-BK interval-valued inference templates

The upper bound of the class *mx* interval-valued SUB-BK inference template is given by

$$(R < S)_{ik} = \underset{j}{\owedge}\left(R_{ij} \begin{pmatrix} & lower \\ \rightarrow & \\ & mx \end{pmatrix} S_{jk}\right), \quad x = 1..5$$

and the lower bound of the class *mx* interval-valued SUB-BK inference template is given by

$$(R < S)_{ik} = \underset{j}{\owedge}\left(R_{ij} \begin{pmatrix} & lower \\ \rightarrow & \\ & mx \end{pmatrix} S_{jk}\right), \quad x = 1..5$$

SUB-B interval-valued inference templates

The upper bound of the class *mx* valued SUB-B inference template is given by

$$(R \triangleleft S)_{ik} = \min\left(\underset{j}{\owedge}\left(R_{ij} \begin{pmatrix} & lower \\ \rightarrow & \\ & mx \end{pmatrix} S_{jk}\right) \underset{j}{\ovee} R_{ij}, \underset{j}{\ovee} S_{jk}\right),$$

x= 1..5

and the lower bound of the class *mx* valued SUB-B inference template is given by

$$(R \triangleleft S)_{ik} = \min\left(\underset{j}{\owedge}\left(R_{ij} \begin{pmatrix} & lower \\ \rightarrow & \\ & mx \end{pmatrix} S_{jk}\right) \underset{j}{\ovee} R_{ij}, \underset{j}{\ovee} S_{jk}\right),$$

x= 1..5

SUB-K interval-valued inference templates

The upper bound of the class *mx* valued SUB-K inference template is given by

$$(R \triangleleft_k S)_{ik} = \underset{j}{\owedge}\left(\owedge\left(R_{ij} \begin{pmatrix} & lower \\ \rightarrow & \\ & mx \end{pmatrix} S_{jk}\right),\ \underset{j}{\ovee}\ (\owedge(R_{ij}, S_{jk}))\right), \qquad x = 1..5$$

and the lower bound of the class *mx* valued SUB-B inference template is given by

$$(R \triangleleft_k S)_{ik} = \bigwedge\Big(\bigwedge_j \big(R_{ij} \left(\overset{lower}{\underset{mx}{\rightarrow}}\right) S_{jk} \big),$$

$$\bigvee_j \big(\bigwedge(R_{ij}, S_{jk})\big)\Big), \qquad x = 1..5$$

where

$$a \left(\overset{upper}{\underset{m1}{\rightarrow}}\right) b = \min(1, 1-a+b),$$

$$a \left(\overset{lower}{\underset{m1}{\rightarrow}}\right) b = (1-a) \vee b$$

$$a \left(\overset{upper}{\underset{m2}{\rightarrow}}\right) b = \min(1, b/a),$$

$$a \left(\overset{lower}{\underset{m2}{\rightarrow}}\right) b = \max(0, (a+b-1)/a)$$

$$a \left(\overset{upper}{\underset{m3}{\rightarrow}}\right) b = (a \wedge b) \vee (1-a),$$

$$a \left(\overset{lower}{\underset{m3}{\rightarrow}}\right) b = \max(a+b-1, 1-a)$$

$$a \left(\overset{upper}{\underset{m4}{\rightarrow}}\right) b = ((1-a) \vee b) \wedge (a \vee (1-b) \vee (b \wedge (1-a))),$$

$$a \left(\overset{lower}{\underset{m4}{\rightarrow}}\right) b = \min(\max(1-a, a+b-1), \max(b, 1-a-b))$$

$$a \left(\overset{upper}{\underset{m5}{\rightarrow}}\right) b = \max(\min(1, b/a), 1-a),$$

$$a \left(\overset{lower}{\underset{m5}{\rightarrow}}\right) b = \max((a+b-1)/a, 1-a)$$

An example of the class m1 upper and lower inference templates of the SUB-K interval-valued inference template are

$$\min\Big(\frac{1}{n}\sum_{j=1}^{n} \big(R_{ij} \left(\overset{upper}{\underset{m1}{\rightarrow}}\right) S_{jk}\big),$$
$$\text{OrBot}\,(\text{AndBot}\,(R_{ij}, S_{jk}))\Big)$$

$$\min\Big(\frac{1}{n}\sum_{j=1}^{n} \big(R_{ij} \left(\overset{lower}{\underset{m1}{\rightarrow}}\right) S_{jk}\big),$$
$$\text{OrBot}\,(\text{AndBot}\,(R_{ij}, S_{jk}))\Big)$$

These upper and lower inference templates of class m1 sub-product type can be instantiated with the following bounded connectives [15] AndBot(a,b) = max(0,a+b-1), AndTop(a,b) = min(a,b), OrBot(a,b) = min(1, a+b) and OrTop(a,b) = max(a,b).

Other inference structures of the class m1 can be found in [29].

7.0 CONCLUSION

Inferencing in an environment where the information is incomplete, inexact and context-sensitive is difficult. Earlier inference structures were point-based. There are numerous possibilities in the choice of the fuzzy connectives in formulating an inference structure of this kind. Due to the fuzzy nature of the information which we are processing, it is important to realize that it is possible to bind the inferred values using point-based inference structures. This is the interval-valued inferencing. Interval-valued inference is becoming popular because it captures the upper and the lower bounds of an inference. The evaluation of the correctness of inference would then be more accurate compared to point-based fuzzy inference. We have shown that these inference structures can be classified and we have provided five measures or classes of interval-valued fuzzy relational inference templates corresponding to the sub-triangle relational product. These templates can be instantiated with a variety of connectives that are bounded.

REFERENCES

[1] L. A. Zadeh, Similarity Relations and Fuzzy Orderings, *Infor. Science*, Vol. 3, 1971, pp. 177-200.

[2] L. A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Tans. Syst. Man, Cyber.*, Vol. 3, 1973, pp. 28-44.

[3] M. Mizumoto, S. Fukami and K. Tanaka, Some methods of fuzzy reasoning, in M. M. Gupta, R. K. Ragade, and R. R. Yager Eds., *Advances in Fuzzy Set Theory and Applications*. Amsterdam. New York. Oxford, 1979.

[4] E. H. Mamdami, Application of fuzzy logic to approximate reasoning. *IEEE Trans. Compu.*, Vol. 26, 1977, pp. 77-88.

[5] J. F. Baldwin, A new approach to approximate reasoning using fuzzy logic. *Fuzzy Sets and Systems,* Vol. 2, 1979, pp. 309-325.

[6] J. F. Baldwin, and B. W. Pilsworth, Axiomatic approach to implication for approximate reasoning with fuzzy logic. *Fuzzy Sets and Systems,* Vol. 3, 1980, pp. 193-219.

[7] J. F. Baldwin, and N. C. F. Guild, Feasible algorithms for approximate reasoning using fuzzy logic. *Fuzzy Sets and Systems,* Vol. 3, 1980, pp. 225-251.

[8] Y. Tsukamoto, An approach to fuzzy reasoning method, in M. M., Gupta, R. K. Ragade, R. R. Yager Eds., *Advances in Fuzzy Set Theory and Applications* North Holland, Amsterdam, 1979, pp. 137-149.

[9] W. Bandler, and L. J. Kohout, *Mathematical Relations, their Products and Generalized Morphisms, Technical Report, Man-Machine Systems Laboratory, EES-MMS-REL 77-3*, Dept. of Electrical Engineering, Univ. of Essex, 1977.

[10] W. Bandler, and L. J. Kohout, Fuzzy relational products and fuzzy implication operators, in: *International Workshop on Fuzzy Reasoning - Theory and Applications, Queen Mary College, University of London, 1978*.

[11] W. Bandler, and L. J. Kohout, Semantics of implication operators and fuzzy relational products, *International Journal of Man-Machine Studies,* Vol. 12, 1980, pp. 89-116.

[12] W. Bandler and L. J. Kohout, Fuzzy power sets and fuzzy implication operators, *Fuzzy Sets and Systems*, Vol. 4, 1980, pp. 13-30.

[13] B. D. DeBaets and E. Kerre, Fuzzy relational composition, *Fuzzy Sets and Systems,* Vol. 60, 1993, pp. 109-120.

[14] D. Dubois, and H. Prade, A review of fuzzy set aggregation connectives. *Information Sciences*, Vol. 36, 1985, pp. 85-121.

[15] W. Bandler and L. J. Kohout, The interrelations of the principal fuzzy logical operators, in M. M. Gupta, A. Kandel, W. Bandler and J. B. Kiszka Eds., *Approximate reasoning in Expert System*, 1985, pp. 767-780.

[16] L. J. Kohout, and W. Bandler, Interval-valued systems for approximate reasoning based on checklist paradigm semantics, in: P. P. Wang Ed., *Advances in Fuzzy Set Theory and Technology*, Vol. 1, 1993, pp. 167-193.

[17] W. Bandler and L. J. Kohout, Relations, mathematical in M. G. Singh Ed., *Systems and Control Encyclopedia*, Oxford, Pergamon Press, 1987, pp. 4000-4008.

[18] R. Willmott, *Transitivity of Inclusion and Equivalence in Fuzzy Power-Set Theory, Report No. FRP-10,* Department of Mathematics, University of Essex, 1981.

[19] Menger, Statistical metrics, *Proc. Nat. Acd. Sci. U.S.A.*, Vol. 28, 1942, pp. 535-537.

[20] R. R. Yager, On a general class of fuzzy connectives, *Fuzzy Sets and Systems,* Vol. 4, 1980, pp. 235-242.

[21] S. Weber, A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms, *Fuzzy Sets and Systems,* Vol. 11, 1983, pp. 115-134.

[22] M. M. Gupta and J. Qi, Theory of T-norms and fuzzy inference methods, *Fuzzy Sets and Systems*, Vol. 40, 1991, pp. 431-450.

[23] L. J. Kohout, J. Anderson and W. Bandler, Clinaid: An Overview, in L. J. Kohout, J. Anderson, and W. Bandler Eds., *Knowledge-based Systems for Multiple Environments*, Aldershot, U.K., Ashgate Publ. Gower, 1992, pp. 173-198.

[24] C. Alsina, On the family of connectives for fuzzy sets, *Fuzzy Sets and Systems*, Vol. 16, 1985, pp. 231-235.

[25] C. Alsina, E. Trillas, and L. Valverde, On some logical connectives and fuzzy set theory, *J. Math. Anal. Appl,*. Vol. 93, 1983, pp. 15-26.

[26] J. Dombi, A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness induced by fuzzy operators, *Fuzzy Sets and Systems,* Vol. 8, 1982, pp. 149-163.

[27] B. Werners, Aggregation models in mathematical programming. In G. Mitra Ed., *Mathematical Models for Decision Support*. Berlin Springer-Verlag, 1988, pp. 295-319.

[28] W. Silvert, Symmetric summation: A class of operations on fuzzy sets, *IEEE Transaction Systems, Man Cybernet*, Vol. 9, 1979, pp. 667-659.

[29] K. M. Yew, *Interval-valued approximate inference using fuzzy relational techniques*, Ph.D Dissertation, Florida State University, 1995.

BIOGRAPHY

Yew Kok Meng is a lecturer in the Faculty of Computer Science and Information Technology, University of Malaya, Malaysia.

L. J. Kohout is a Professor in the Department of Computer Science, Florida State University, United States of America.